COL7160 : Quantum Computing

Lecture 12: Quantum Fourier Transform

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1 Overview

In this lecture, we study the *Discrete Fourier Transform (DFT)* and its quantum analogue, the *Quantum Fourier Transform (QFT)*. The Fourier transform plays a central role in quantum algorithms such as phase estimation. We first review the linear-algebraic structure of the DFT, show that it is unitary, and then explain how it can be implemented efficiently on a quantum computer using only polynomially many quantum gates.

2 Discrete Fourier Transform

Let $N \in \mathbb{N}$ and define

$$\omega_N = e^{2\pi i/N},$$

which is an N-th root of unity.

Definition 1. The *Discrete Fourier Transform matrix* $F_N \in \mathbf{C}^{N\times N}$ is defined by

$$(F_N)_{j,k} = \frac{1}{\sqrt{N}}\,\omega_N^{jk}, \quad j,k \in \{0,1,\ldots,N-1\}.$$

The k-th column of F_N is given by

$$|(F_N)_k\rangle = \frac{1}{\sqrt{N}}\begin{pmatrix} 1 \\ \omega_N^k \\ \omega_N^{2k} \\ \vdots \\ \omega_N^{(N-1)k} \end{pmatrix}.$$

2.1 Roots of Unity

The roots of unity satisfy the following properties:

- $\omega_N^k = 1$ if and only if $k \equiv 0 \pmod N$.
- $\sum_{j=0}^{N-1} \omega_N^{jx} = 0$ if $x \not\equiv 0 \pmod N$.

The last identity follows from the geometric series formula:

$$\sum_{j=0}^{N-1} \omega_N^{jx} = \frac{\omega_N^{xN} - 1}{\omega_N^x - 1}.$$

2.2 Unitarity of the Fourier Matrix

Theorem 2. *The matrix F_N is unitary.*

Proof. We show that the columns of F_N form an orthonormal set. Consider two columns k_1 and k_2:

$$\langle (F_N)_{k_1} | (F_N)_{k_2}\rangle = \frac{1}{N}\sum_{j=0}^{N-1} \omega_N^{j(k_2-k_1)}.$$

If $k_1 = k_2$, the sum equals 1. Otherwise, $k_2 - k_1 \not\equiv 0 \pmod N$ and the sum vanishes by the roots-of-unity identity. Hence the columns are orthonormal, and $F_N^\dagger F_N = I$. $\square$

3 Fourier Transform on Quantum States

Since F_N is unitary, it corresponds to a valid quantum operation. For a quantum state

$$|v\rangle = \sum_{x=0}^{N-1} v_x |x\rangle ,$$

the Fourier transform acts as

$$|v\rangle \longmapsto F_N |v\rangle .$$

However, measuring the output state does not directly reveal all Fourier coefficients, since measurement yields only one basis outcome. Thus, the QFT is powerful only when combined with additional algorithmic structure.

4 Phase Gates

Before constructing the QFT circuit, we define the phase gates used in the construction.

4.1 Single Qubit Phase Gate

The phase gate $R(\theta)$ is defined as:

$$R(\theta) = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\theta} \end{pmatrix}$$

4.2 Controlled Phase Gate

The controlled-phase gate applies a phase only when both the control and target qubits are in the state $|1\rangle$.

$$P(\theta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & e^{i\theta} \end{pmatrix}$$

Remark 3 (Symmetry). For the controlled phase gate, it does not matter which bit is the control bit and which is the target bit. The gate applies the phase $e^{i\theta}$ if and only if both bits are 1 ($|11\rangle$).

This symmetry allows us to write the gate simply as a vertical line connecting two qubits with dots, without explicitly specifying a "direction".

5 Quantum Fourier Transform (QFT)

Let $N = 2^n$. The Quantum Fourier Transform is the unitary operation that maps a computational basis state $|x\rangle$ to:

$$|x\rangle \xrightarrow{QFT} \frac{1}{\sqrt{N}} \sum_{y=0}^{N-1} \omega_N^{xy} |y\rangle , \quad \text{where } \omega_N = e^{2\pi i/N}$$

5.1 Product State Representation

We can expand the state to see that the QFT produces a tensor product of single-qubit states. Let the binary representations be $x = x_1 x_2 \ldots x_n$ and $y = y_1 y_2 \ldots y_n$.

$$y = \sum_{j=1}^{n} y_j 2^{n-j}$$

Substituting this into the definition:

$$
\begin{aligned}
|x\rangle \mapsto \frac{1}{\sqrt{N}} \sum_{y=0}^{N-1} \omega_N^{x \sum_{j=1}^{n} y_j 2^{n-j}} |y\rangle \\
= \frac{1}{\sqrt{N}} \sum_{y=0}^{N-1} \left(\prod_{j=1}^{n} \omega_N^{x y_j 2^{n-j}} \right) |y\rangle \\
= \frac{1}{\sqrt{N}} \bigotimes_{j=1}^{n} \left(|0\rangle + \omega_N^{x 2^{n-j}} |1\rangle \right)
\end{aligned}
$$

5.1.1 Analyzing the Phase

Consider the exponent in the phase term $\omega_N^{x2^{n-j}} = e^{2\pi i \frac{x2^{n-j}}{2^n}} = e^{2\pi i \frac{x}{2^j}}$. Using the binary expansion $x = \sum_{l=1}^{n} x_l 2^{n-l}$, we can write:

$$
\frac{x}{2^j} = x_1 \cdots x_{n-j}.x_{n-j+1} \cdots x_n
$$

The integer part $(x_1 \ldots x_{n-j})$ contributes $e^{2\pi i K} = 1$ to the phase and can be removed. We are left with only the fractional part:

$$
0.x_{n-j+1} \cdots x_n = \sum_{k=n-j+1}^{n} \frac{x_k}{2^{j-(n-k)}}
$$

Thus, the QFT state can be written as:

$$
|x\rangle \xrightarrow{QFT} \frac{1}{2^{n/2}} \left(|0\rangle + e^{2\pi i (0.x_n)} |1\rangle \right) \otimes \left(|0\rangle + e^{2\pi i (0.x_{n-1} x_n)} |1\rangle \right) \otimes \cdots \otimes \left(|0\rangle + e^{2\pi i (0.x_1 \cdots x_n)} |1\rangle \right)
$$

6 Circuit Construction

We need a proper order to perform these updates. Notice that the phase of the j-th qubit depends on the input bits $x_j, \ldots, x_n$.

- The Hadamard gate H on $|x_j\rangle$ creates the superposition $|0\rangle + e^{2\pi i (0.x_j)} |1\rangle$.
- The controlled phase gates $P(\theta)$ add the remaining terms $0.0x_{j+1} \cdots x_n$ to the phase.

Since the output $|z_1\rangle$ (which corresponds to the phase $0.x_1 \ldots x_n$) depends on all input bits, while $|z_n\rangle$ depends only on x_n, we process the qubits in reverse or include SWAP gates at the end.

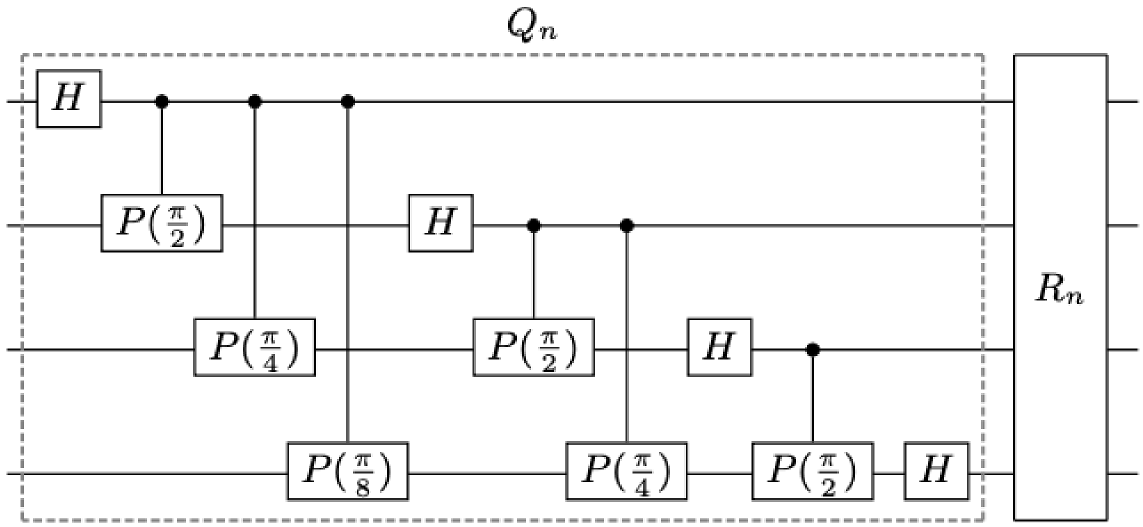

Figure 1: Quantum Fourier Transform circuit for three qubits, including controlled phase rotations and final swap gates R_n.

6.1 Gate Complexity

The total number of gates is $O(n^2)$:

- Hadamard Gates: n gates (one per qubit).
- Controlled Phase Gates: n distinct gates. Each gate is applied at most n times.
- Swap Gates: $\frac{n}{2}$ swap gates are required at the end to reverse the order of the qubits to match the standard binary representation.

7 Exercises

1. Read about the classical Fast Fourier Transform (FFT)
2. Think about the full quantum phase estimation algorithm using the QFT.
3. Think the relationship between integer factorization and order finding.